Weak measurements and the two state vector formalism

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The aim of these four lectures is to make you familiar with the two state vector formalism and weak measurements. I will try to keep the physical motivation alive throughout the lectures. The first three lectures will include the mathematical formalism used, with examples mixed in. I will explicitly derive a strong and weak measurement procedure using a qubit meter. This is unlike the majority of literature where the meter is a continuous variable system. I feel that the qubit derivation is simpler and the formalisms are very similar. I will however describe experimental implantations that are usually done with continuous degrees of freedom. I will also avoid discussion of mixed states and generalized two-state vectors.

The lectures are planned as follows: In the first lecture I will discuss the two state vector formalism. In the second lecture I will go through the von Neumann measurement and the weak measurement schemes for projection operators. In the third lecture I will derive the weak measurement procedure in a more general way and discuss some of the properties of weak measurements. The final lecture will be used for discussing theoretical and experimental work done in the last decade (and a bit).

I will cite the relevant literature as I go, at this point I will mention a few reviews. The first is my personal choice for an introduction to weak measurements of the two state vector formalism written by two of the originators of these ideas, Yakir Aharonov and Lev Vaidman [1]. There are, as far as I know, three more recent reviews on weak measurements [2–4] another sources is a book on quantum paradoxes [5] which has some related material. I should also mention the original papers [6, 7].

1. Assignments

I will give a number of exercises during the lectures, these should be submitted after the course by anyone registered. I suggest that you do each exercise before the next lecture, they should not be be very time consuming but will hopefully help build some intuition.
Part I

The two state vector formalism

I. MOTIVATION- ASKING A PHOTON ABOUT ITS PAST

To motivate the possibly dry derivation that will follow, I will describe a recent experiment [8] which yields somewhat strange results. The aim of the next two lectures is to give simple tools to understand this experiment or at the very least calculate the results in a simple way.

The experiment is a type of which-path optical experiment, apart from the light source (a laser) there are three components that may require some explanation (more details will come in the next lecture).

- Beam splitter (BS) - This is a partially reflecting mirror. Photons going in will come out in a superposition of going being reflected and transmitted. The amplitudes depend on the particular setup.

- Piezo-electrically driven mirror (PZM) - this mirror shakes (changes angle) very slightly at some frequency, reflected light will be slightly displaced according the the angle of the mirror.

- Quad cell photo detector - This is a photon detector that is sensitive to the displacement induced by the PZM. Fourier transforming the output signal (after many counts) gives a spectrum of the frequency of displacement.

The experiment has two interferometers, one nested inside the other. The five mirrors in the interferometers (labled A,B,C,E,F) are PZMs set at different frequencies. When a photon hits a mirror it leaves a trace - displacing the photon slightly. If the photon makes it to the detector its position is recorded.

The spectrum should tell us the story of the photon’s travel, each time it passes a mirror it gets a ‘stamp’ at a different frequency, however looking at the final spectrum we get something strange. Supposedly the photon hit mirrors A,B and C but not E,F. This is strange since any photon hitting A and/or B should enter through e and exit through f, however there is no trace of this path.
FIG. 1: A balanced interferometer nested inside a 2:1 interferometer. Only photons arriving at the detector D are counted in the experiment. A weak measurement of the photon’s past shows that it interacted with mirrors $A, B, C$ but not $E, F$.

To explain this strange behavior we look at two ingredients of the experiment. 1. Weak measurement - The initial width of the photon is much larger than the amplitude of the displacement induced by the PZMs. 2. Non trivial post-selection - Many photons do not make it to the detector, rather they go to the second output port of the last BS but the measured signal is only from those that did make it.

Today we will discuss this second element, post-selection, and see some interesting consequences. Experiments with post-selection have a more intuitive explanation in the two state vector formalism.
II. NOTATION - PROJECTIVE MEASUREMENTS AND OBSERVABLES

In this course we will discuss measurements of observables, before going into any details it is important to establish some consistent notation.

A. Observables

An observable is an Hermitian operator $A$ with eigenvalues $a_i$ and eigenvectors $|i_a\rangle$. In some cases (most of the interesting ones to follow) the observables are degenerate, i.e a number of eigenvectors $|i_a\rangle$ have the same eigenvalue $a_i = a_l$. We denote $A_l$ as the projector onto the degenerate subspace

$$A_l = \sum_{i|a_i=a_l} |i_a\rangle \langle i_a|$$

so

$$A = \sum a_l A_l$$

where the sum is over the disjoint subspaces.

The eigenvalues $a_l$ are real numbers, their precise value is usually not important since for most practical purposes we can rescale Hermitian operators by adding a constant multiple of the identity and/or multiplying by a constant. Degeneracy on the other hand will play a vital role for measurements. In most of the discussion we will limit ourselves to two types of operators: a) Dichotomic - Observables with two eigenvalues (often rescaled to $\{0, 1\}$ i.e projection operators); b) Observables with integer eigenvalues.

B. Measurements

Our approach to measurement will be similar to the one in standard QM textbooks. Measurements have outcomes that correspond to the eigenvalues of the corresponding observables. In the next lecture I will discuss the measurement process in more detail. For now we will only describe some features of ‘standard’ quantum projective measurements which I will call strong measurements.

• A strong measurement of the observable $A$ on a system in the state $|\psi\rangle$ will produce
the outcome \( a_l \) with probability given by the Born rule

\[
P(a_l | A, \psi) = \langle \psi | A_l | \psi \rangle
\]  

(2)

- Following a strong measurement of \( A \) with outcome \( a_l \) the state of the system will be projected into the corresponding subspace. i.e a system initially in the state \( |\psi\rangle \) will be projected into the state

\[
\frac{1}{\sqrt{\langle \psi | A_l | \psi \rangle}} A_l |\psi\rangle = \frac{1}{\sqrt{P(a_l | A, \psi)}} A_l |\psi\rangle
\]  

(3)

This is often called the ‘wave function collapse’ although this terminology is sometimes frowned upon. I will use the term ‘collapse’ once or twice for simplicity, but this by no means implies any specific interpretation of the measurement process.

Although these features are not general (i.e there are other types of measurements) there is a point which should be stressed: For our purposes any measurement on system \( S \) will have an associated outgoing (post measurement) state of a composite system that includes \( S \). In the case of a strong measurement of a non-degenerate observable the post-measurement state of \( S \) will be pure. I will discuss this in more detail in the next lecture.

In some cases we are interested in the expectation value of an observable \( \langle \psi | A | \psi \rangle \) which has the interpretation of the statistical average of many measurements made on identically prepared systems. A projection operator is an Hermitian operator with eigenvalues either 0 or 1. By the Born rule (2) the expectation value of a projection operator is the probability of the outcome ‘1’. Any dichotomic observable can be re-scaled into a projection operator.

### III. THE TWO STATE VECTOR FORMALISM

The Schrodinger equation is time symmetric, although we are used to setting our boundary conditions in the past (a preparation or pre-selection) there is no reason (in theory) why we cannot set our boundary condition in the future (a post-measurement or post-selected state). This is true for classical Newtonian physics as well. In Newtonian physics setting one boundary condition will uniquely determine the other and there is no gain in using both. In quantum theory on the other hand one boundary condition does not uniquely determine the other. To clarify this statement we will introduce an operational picture.
We consider the following type of experiment\(^1\): A system \(S\) is pre-selected (prepared) in a state \(|\psi\rangle\) at time \(t = 0\). It then evolves and possibly interacts with some external system (a measurement) finally at time \(t_f\) a strong measurement of a non-degenerate observable \(M_f\) is made such that the post-selected (post-measurement) state is \(|\phi\rangle\) (an eigenstate of \(M_f\)).

Generally the final measurement \(M_f\) can have many different outcomes that are compatible with the forward evolving state just before \(t_f\). Moreover different choices of \(M_f\) will produce different sets of possibly compatible states. So even when the intermediate dynamics of the system are unitary (i.e deterministic) there are many compatible post-selected states. One would therefore expect to have more information about the intermediate state of the system if both boundary conditions are included.

The description of the system in the intermediate time \(0 < t < t_f\) will be as a two-state vector \(\langle \phi | \psi \rangle\) where \(|\psi\rangle\) is the pre-selected state evolving forward in time and \(\langle \phi |\) is the post-selected state evolving backwards in time.

\[P(a_i|\phi f, \psi_0, A) = |\langle \phi | A_i |\psi \rangle|^2\]

\(^1\) This model is somewhat restricted to have pure pre and post selection but it can be generalized[1]
and the denominator of (4) is

\[ P(\phi_f|\psi_0, A) = \sum_i P(a_i, \phi_f|\psi_0, A) = \sum_i |\langle \phi | A_i |\psi \rangle|^2 \] (9)

Unless explicitly stated we will from now on assume everything is conditioned on our pre and post selection. Probabilities for outcomes of a measurement will always be conditioned on that measurement being performed. So

\[ P(a_i) = \frac{|\langle \phi | A_i |\psi \rangle|^2}{\sum_i |\langle \phi | A_i |\psi \rangle|^2} \] (10)

This is the ABL formula for probabilities of strong measurements in the two state vector formalism [6]. For a generalization of this formula using generalized two state vectors and mixed states see [1].

**Exercise 1.** Given a projection operator \( A = (1)A_{yes} + (0)A_{no} \) what is the probability of the result 'yes' when the pre and post selection are the same \( |\psi\rangle = |\phi\rangle \)? What is the condition for \( P(\text{yes}) = 1 \)?

**IV. THE THREE BOX PARADOX**

Imagine the following scenario [9]: An experimenter, Alice, has a quantum ball that can be in one of three boxes \( A, B, C \). At time \( t_i \) she prepares the state of the ball in a superposition of all three boxes. \( |\psi\rangle = \frac{1}{\sqrt{3}}[|A\rangle + |B\rangle + |C\rangle] \) she then leaves the room. After some time she returns and, by making a projective measurement \( M_f \), finds the state to be \( |\phi\rangle = \frac{1}{\sqrt{3}}[|A\rangle + |B\rangle - |C\rangle] \).

Alice’s friend Bob then tells her that he opened box A. What is the probability that he found a ball in this box?

To simplify the calculation let us first calculate the probability of Bob not finding a ball in box A conditioned on Alice succeeding in post-selecting \( |\phi\rangle \). The corresponding projection operator is \( A_{NO} = |B\rangle \langle B| + |C\rangle \langle C| \) using (8) we have

\[
P(a_{no}, \phi_f|\psi_0, A) = \frac{1}{9} |[|A\rangle + \langle B| + \langle C'| |B\rangle \langle B| + |C\rangle \langle C| [\langle A\rangle + |B\rangle + |C\rangle]|] = 0 \quad (11)
\]

\[
= \frac{1}{9} |[\langle B| - \langle C'|]|B\rangle + |C\rangle]| = 0 \quad (12)
\]
The probability of finding a ball is given by ABL (10)

\[ P(a_{\text{yes}}) = \frac{\left| \langle \phi | A_{\text{yes}} | \psi \rangle \right|^2}{\left| \langle \phi | A_{\text{yes}} | \psi \rangle \right|^2 + 0} = 1 \] (13)

Where \( A_{\text{yes}} = |A\rangle \langle A| \)

From symmetry we can see that \( P(B_{\text{yes}}) = P(A_{\text{yes}}) = 1 \) so the following statements can be made

- Had Alice opened box A she would have found a ball
- Had Alice opened box B she would have found a ball
- Had Alice opened box C she might have found a ball

**Exercise 2.** Calculate \( P(c_{\text{yes}}) \)

This leads to the following counterfactual reasoning

- There was a ball in box A
- There was a ball in box B
- There might have been a ball in box C

Obviously these statements do not make sense together, the result depends on the measurement. Moreover the fact that the results are ‘strange’ depends on the fact that the ’0’ eigenvalue is degenerate.

However these kinds of statements cannot be made in a ‘one state vector’ scenario where probabilities are expectation values of projection operators. Expectation values add up so

\[ \text{Exp}(A_{\text{yes}}) + \text{Exp}(B_{\text{yes}}) + \text{Exp}(C_{\text{yes}}) = \text{Exp}(A_{\text{yes}} + B_{\text{yes}} + C_{\text{yes}}) = \text{Exp}(\mathbb{1}) = 1. \]
Part II

Weak measurements

‘There is no free lunch’, but as any grad student knows sometimes lunch is free for all practical purposes.

Quantum measurements have an inherent disturbing property, we can’t get information without changing the state. But what can we do at the limit of no disturbance? Can we still get some information out? Not really but...

V. VON NEUMANN MEASUREMENTS

We will begin with a qubit variant of the von Neumann measurement scheme for strong projective measurements. To measure an observable \( A \) on system \( S \) we need to couple it to another system called a measurement device or meter \( M \) with a pointer variable \( Q \) and a conjugate momentum \( P \).

Before getting into the details of how the measurement of a dichotomic observable works, I will sketch the measurement of a general non-degenerate observable \( A \) with integer eigenvalues \( \{0..d-1\} \). The meter will be a quasi-quantum \( d \) dimensional object. It is quasi-quantum in the sense that it can be observed classically in the \( Q \) basis. The initial state of the system \( S \) in the eigenbasis of \( A \) is \( \sum_{i=0}^{d-1} \lambda_i \ket{i_a} \) and \( M \) is prepared in the \( \ket{0} \) state. The measurement interaction will create the following state

\[
\sum_{i=0}^{d-1} \lambda_i \ket{i_a}^S \ket{i}^M
\]  

A classical observation would ‘collapse’ the state into a classical result \( j \) and the corresponding eigenstate of \( A \), \( \ket{j_a}^S \). This is a very general sketch of what we want from a strong measurement. The process before the classical observation (which has no dynamical model) is sometimes called the ‘pre-measurement’. In the weak measurement literature the term measurement is used to describe this pre-measurement or interaction stage. I will comment on this later.
1. The von Neumann scheme

I will now describe a qubit variant of the scheme originally introduced by von Neumann. Since the ‘collapse’ stage of the measurement has no dynamical model, the aim is to describe the so-called ‘pre-measurement’ stage which leads to a state of the type (14).

In our qubit scenario we can choose our pointer $Q = \sigma_z$ and the conjugate momentum $P = \sigma_y$ \(^2\) Since our meter is a two level system we will use it to measure an observable $A$ with only two eigenvalues $\{0, 1\}$ (a projector).

$$A = A_+ + 0 A_0$$

(15)

Where $A_+$ and $A_0$ are projectors onto the relevant subspaces.

The coupling between the system and the meter is very short and can be described by the Hamiltonian

$$H_i = f(t) A^S \sigma_y^M$$

(16)

where $f(t)$ is very large for a very short period of time $\int_{t_i-\epsilon}^{t_i+\epsilon} f(t) = g$. The rest of the Hamiltonian can be neglected for the time of interaction so the evolution is

$$U_i = e^{-ig A^S \sigma_y^M}$$

(17)

A system-meter state initially

$$|SM_0\rangle = |\psi\rangle^S |0\rangle^M$$

(18)

will evolve to

$$|SM_{t_i+\epsilon}\rangle = e^{-ig A^S \sigma_y^M} |\psi\rangle^S |0\rangle^M = A_+ |\psi\rangle e^{-ig \sigma_y^M} |0\rangle + A_0 |\psi\rangle e^{-i0 \sigma_y^M} |0\rangle$$

(19)

now choosing $g = \pi/2$ and using $e^{-ig \sigma_y} = \cos(g) 1 - i \sin g \sigma_y$ and $\sigma_y |0\rangle = i |1\rangle$ we get

$$|SM_{t_i+\epsilon}\rangle = A_+ |\psi\rangle |1\rangle + A_0 |\psi\rangle |0\rangle$$

(20)

\(^2\) for a qubit meter any anti-commuting pair of Pauli operators can be the pointer and momentum.
A. What about post-selection?

In principle at this point the meter would somehow ‘collapse’ into one of the eigenstates, however if there is no further interaction the exact timing of this ‘collapse’ is not relevant and can be delayed. The practical thing to do would be to consider a collapse (or at least dephasing) at this point. However when we consider post-selection it is better to leave the system coherent (as it would in the many worlds interpretation).

If now we post-select the system in the state $|\phi\rangle$. We get

$$|SM_{t_f}\rangle \propto \langle \phi | A_+ | \psi \rangle |1\rangle + \langle \phi | A_0 | \psi \rangle |0\rangle$$

(21)

Normalizing we get

$$|SM_{t_f}\rangle = \frac{\langle \phi | A_+ | \psi \rangle |1\rangle + \langle \phi | A_0 | \psi \rangle |0\rangle}{\sqrt{\langle \phi | A_+ | \psi \rangle^2 + \langle \phi | A_0 | \psi \rangle^2}}$$

(22)

We can see that this leads to the ABL rule.

Exercise 3. Derive the ABL rule for a dichotomic observable using the von Neumann measurement scheme.

Exercise 4. Derive the ABL rule for a non-degenerate observable using the post-interaction state (14).

VI. WEAK MEASUREMENTS

A. The weak value

A weak measurement is essentially a von Neumann measurement interaction (17) with very small $g$. Before going into technical details it may help to get some intuition about the expected result. Let us consider a case where the interaction between a meter and the system is so weak that post-selecting the original state is extremely likely. The readout on the meter is going to be very fuzzy so we have to repeat the experiment many times on an ensemble of identically prepared state and get some statistical result. What would we expect to be the average reading?

If we think of the fuzzyness as a type of noise that averages out, we should expect the average meter reading to be close to the expectation value. Indeed this is what we get with
a weak measurement. If we post-select, however we will get a weak value

$$\{A\}_w = \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle}$$

(23)

Note that with $|\phi\rangle = |\psi\rangle$ this is the expectation value.

**B. The weak measurement of a projection operator**

Warning: The dynamical process below is that of an interaction with a meter followed by post selection and a readout. There have been some objections to the name ‘measurement’ as a description of this process. It may be helpful to think of this as a weak ‘pre-measurement’, rather than a type of measurement.

We will start with a weak measurement of a dichotomic variable. Unlike the case of strong measurement we will now have the meter in the $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ state so,

$$|SM_0\rangle = |\psi\rangle_S |+\rangle^M$$

(24)

will evolve to

$$|SM_{t+i\epsilon}\rangle = e^{-igA^S_\sigma^M_y} |\psi\rangle^S |+\rangle^M$$

(25)

$$= A_+ |\psi\rangle e^{-ig\sigma^M_y} |+\rangle + A_0 |\psi\rangle e^{-i\theta\sigma^M_y} |+\rangle$$

(26)

$$= A_+ |\psi\rangle (\cos(g) - i \sin(g) \sigma_y) |+\rangle + A_0 |\psi\rangle |+\rangle$$

(27)

$$= \cos(g) |\psi\rangle |+\rangle + [1 - \cos(g)] A_0 |\psi\rangle |+\rangle - i \sin(g) A_+ |\psi\rangle \sigma_y |+\rangle$$

(28)

post selecting gives the unnormalized state,

$$\sqrt{P(\phi)} |M_{t+i}\rangle = \cos(g) \langle \phi | \psi \rangle |+\rangle + [1 - \cos(g)] \langle \phi | A_0 | \psi \rangle |+\rangle - i \sin(g) \langle \phi | A_+ | \psi \rangle \sigma_y |+\rangle$$

(29)

$$= \langle \phi | \psi \rangle \left[ \cos(g) \mathbb{1} + \frac{[1 - \cos(g)] \langle \phi | A_0 | \psi \rangle \mathbb{1} - i \sin(g) \langle \phi | A_+ | \psi \rangle \sigma_y}{\langle \phi | \psi \rangle} \right] |+\rangle$$

(30)

The factor $P(\phi)$ is the the probability of successful post selection We now take the limit $g << 1$ so $\cos(g) \approx 1$ and $\sin(g) \approx g$
\[
\sqrt{P(\phi)} |M_{tf}\rangle \approx \langle \phi | \psi \rangle \left[ \mathbb{I} + \frac{0 \langle \phi | A_0 | \psi \rangle \mathbb{I} - ig \langle \phi | A_+ | \psi \rangle \sigma_y}{\langle \phi | \psi \rangle} \right] |+\rangle \\
= \langle \phi | \psi \rangle \left[ \mathbb{I} - ig \langle \phi | (0) A_0 + A_+ | \psi \rangle \sigma_y \right] |+\rangle \\
= \langle \phi | \psi \rangle \left[ \mathbb{I} - ig \{A_w\} \sigma_y \right] |+\rangle 
\]

(31)

We now normalize using \( \sigma_y |+\rangle = -i |-\rangle \), neglecting terms of order \( g^2 \) we get

\[
P(\phi) \approx | \langle \phi | \psi \rangle |^2 
\]

(34)

We now have (up to terms of order \( g^2 \))

\[
|M_{tf}\rangle \approx [\mathbb{I} - ig \{A_w\} \sigma_y] |+\rangle 
\]

(35)

The readout is the expectation value

\[
\langle M_{tf} | \sigma_z | M_{tf} \rangle \approx \langle + | [\mathbb{I} + ig \{A\}^*_w \sigma_y] \sigma_z [\mathbb{I} - ig \{A_w\} \sigma_y] |+\rangle \\
= \langle + | [\mathbb{I} + ig \{A\}^*_w \sigma_y] [\sigma_z - g \{A_w\} \sigma_x] |+\rangle \\
\approx \langle + | \sigma_z |+\rangle - g \{A\}^*_w \langle + | \sigma_x |+\rangle - g \{A\}_w \langle + | \sigma_x |+\rangle \\
= -2g Re(\{A\}_w) 
\]

(36)

So the pointer’s expectation value is proportional to the real part of the weak value. Since \( g \) is very small we would need to repeat this measurement a large number of times to get the right result. We will come back to this point.

What about the imaginary part? To get the imaginary part we need to change the readout to the expectation value of the conjugate momentum \( \sigma_y \).

\[
\langle M_{tf} | \sigma_y | M_{tf} \rangle \approx \langle + | [\mathbb{I} + ig \{A\}^*_w \sigma_y] \sigma_y [\mathbb{I} - ig \{A_w\} \sigma_y] |+\rangle \\
= \langle + | [\mathbb{I} + ig \{A\}^*_w \sigma_y] [\sigma_y - ig \{A_w\} \mathbb{I}] |+\rangle \\
\approx \langle + | \sigma_y |+\rangle + ig \{A\}^*_w \langle + | \mathbb{I} |+\rangle - ig \{A\}_w \langle + | \mathbb{I} |+\rangle \\
= 2g Im(\{A\}_w). 
\]

(40)

Warning: In general it is always possible to set \( g \) to be small enough for the above approximation to work, unless \( \langle \phi | \psi \rangle = 0 \) in which case the approximation is invalid. Weak values are not defined in this situation!
Not much changes in the derivation above if \(A\) is an observable with more than two eigenvalues. I will repeat a slightly modified version of this calculation in the next lecture to convince you.

VII. THE THREE BOX PARADOX

In the previous lecture we looked at the three box paradox. Let me repeat the main idea. Alice has a quantum ball that can be in one of three boxes \(A, B, C\). At time \(t\), she prepares the state of the ball in a superposition of all three boxes. \(|\psi\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle + |C\rangle\) she then leaves the room. After some time she returns and, by making a projective measurement \(M_f\), finds the state to be \(|\phi\rangle = \frac{1}{\sqrt{3}}(|A\rangle + |B\rangle - |C\rangle\).

Bob then enters the room and tells Alice he looked inside one of the boxes, what is the probability he found a Ball?

- If he looked in box A he definitely found a ball.
- If he looked in box B he definitely found a ball.
- If he looked in box C he found a ball with probability that you calculated.

What if he made a weak measurement?

\[
\begin{align*}
\{A\}_w &= \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle} = \frac{1/3}{1/3} = 1 \\
\{B\}_w &= \frac{\langle \phi | B | \psi \rangle}{\langle \phi | \psi \rangle} = \frac{1/3}{1/3} = 1 \\
\{C\}_w &= \frac{\langle \phi | A | \psi \rangle}{\langle \phi | \psi \rangle} = \frac{-1/3}{1/3} = -1
\end{align*}
\]

If we try to give this a probabilistic interpretation we can say there was a negative number of balls in box C.

A. Weak values for projection operators with definite outcomes

One property of weak values of projection operators is apparent in this scenario: if the outcome of a weak measurement is certain the weak value and strong value coincide.

Theorem 1. The strong measurement of a dichotomic operator will produce a definite outcome if and only if the weak value is an eigenvalue
Proof. Starting with a dichotomic observable \( A = a_1A_1 + a_2A_2 \) where \( A_1, A_2 \) are projectors onto disjoint subspaces and \( A_1 + A_2 = \mathbb{1} \). We have
\[
\langle \phi | \psi \rangle = \langle \phi | A_1 | \psi \rangle + \langle \phi | A_2 | \psi \rangle
\]
(44)

Using ABL (10) we have
\[
P(a_1) = \frac{|\langle \phi | A_1 | \psi \rangle|^2}{|\langle \phi | A_1 | \psi \rangle|^2 + |\langle \phi | A_2 | \psi \rangle|^2}
\]
(45)
assuming \( \langle \phi | \psi \rangle \neq 0 \) (remember weak values are not defined when this assumption fails) we get \( P(a_1) = 1 \leftrightarrow \langle \phi | A_2 | \psi \rangle = 0 \) by (44) this means \( \langle \phi | \psi \rangle = \langle \phi | A_1 | \psi \rangle \) so \( \{ A \}_w = a_1 \) we can use the same reasoning for \( a_2 \).

\[\square\]

VIII. EXPERIMENT: THE PHOTON’S PAST

I will leave the complete explanation of the protocol as a homework exercise. I will go through each component to help form a picture of what’s going on. First I will explain how interferometers work, this may come in handy later. Next I will explain how the whole experiment works with weak measurements.

A. Interferometers

My explanation of interferometers may be slightly different than what you expect in a lab. For simplicity I will compensate for some of the phases so some \( i \)’s might be missing. The overall picture is correct.

1. Beam splitters

A 50:50 beam splitter (fig 2 with \( R : T = 1 : 1 \) is similar to a Hadamard gate. Note that by choosing the orientation we can select which input port corresponds to a \(|0\rangle\).

2. Balanced interferometer

A balanced interferometer is set up in such a way that each input port has a corresponding output port (fig. 3).
FIG. 2: A beam splitter is a simple device with two input ports and two output ports. Each input has an associated reflection and transmission probability $R$ and $T$. For simplicity the beam splitter is described by the ratio $T : R$. The device is not symmetric, the darker part in the picture gives a $-1$ phase to reflected light, while the white part does not.

FIG. 3: A balanced interferometer (with 50:50 Beam splitters) is setup in such a way that light coming in from the top of the first interferometer would end up going out of the bottom of the second.

3. 2:1 interferometer

The external 2:1 interferometer is set up so that photons entering from the top will come out on the right (fig 4).
FIG. 4: An interferometer with 2:1 beam splitters setup in such a way that light coming in from the top of the first interferometer would end up going out of the right of the second.

B. the weak measurement

One way to implement weak measurements in optics is by coupling two modes of the photon, one is the system and the other is the meter. In this experiment the system is the path while the meter is a deflection angle this path. Post selection is done by having a detector at the end we want to post select on. i.e photons reaching the detector are in the desired post-selected state.

For the measurement to be weak the initial width of the beam is large compared to the deflaction. The deflection is given by the mirrors. Each mirror oscillates at a different frequency and the deflection is minimal and oscillates with this frequency.

1. The experiment

In the experiment (fig 5) photons start by going through a 2:1 interferometer (fig 4) but inside it there is a balanced interferometer (fig 3). Any photon going into the nested interferometer should not end up at the detector. The detector acts as the post-selection i.e any photon reaching the detector is in the right post-selected state. Statistical analysis of many hits on the detector can give a spectrum. Non zero weak values are found for A, B, C while \( \{E\}_w = \{F\}_w = 0 \). To see this we note that the pre-selected photons can never reach mirror F. The post selected photons could never have come from E. However, both pre and post selections are compatible with photons going through the nested interferometer.
FIG. 5: A balanced interferometer nested inside a 2:1 interferometer. Any photon entering the nested interferometer should not end up at the detector. The same applies to the backwards traveling wave. Surprisingly the weak values of $A$ and $B$ are not zero.

**Exercise 5.** *Calculate the weak values for the three middle arms $(A,B,C)$*. Hint: *Think of the three box paradox*
Part III

General weak measurements and more paradoxes

In this lecture I will generalize the formalism introduced last time. I will use a slightly different derivation which will allow me to highlight some features of weak measurements. Like the previous derivation, this one requires a few (weak measurement) approximations. As usual the validity of such approximations will depend on the correction terms. I will give some intuition as to why these correction terms can be small, but I do not plan to go into a detailed analysis. In a practical (concrete) case it is always possible to simulate the experiment for different values of the coupling and find the most suitable range of values.

IX. GENERAL WEAK MEASUREMENTS

In the previous lecture I showed how a weak coupling between a quantum system and a meter can give the weak value of a dichotomic observable. In this lecture I will show the following result.

A system $S$ pre-selected in the state $|\psi\rangle$ and post-selected in the state $|\phi\rangle$ is coupled weakly to a (discrete) meter $M$ via the Hamiltonian

$$H_i = f(t)A^S P^M$$

at an intermediate time $t_i$. For weak enough coupling the meter’s dynamics can be described by an effective (possibly non Hermitian) Hamiltonian

$$H_{\text{eff}} \approx f(t)\{A\}_w P^M$$

where

$$\{A\}_w = \frac{\langle \phi | A |\psi\rangle}{\langle \psi | \phi \rangle}$$

is the complex weak value.
A. The weak measurement approximation

In the following derivation I will use the following series expansion for an exponential of a bounded Hermitian operator $B$ twice

$$e^{\alpha B} = 1 + \alpha B + \sum_{k=2}^{\infty} \frac{1}{k!} \alpha^k B^k$$

$$= 1 + \alpha B + O(\alpha^2)$$  \hfill (49)

since $B$ is bounded we can always find $\alpha$ such that the terms $O(\alpha^2)$ are small enough to be neglected.

B. The procedure

At time $t = 0$ the system is prepared in the state $|\psi\rangle$ and the meter is prepared in some arbitrary state $|M_0\rangle$. At time $t_i$ the Hamiltonian (46) is switched on so that the evolution is

$$e^{-igASPM} |\psi\rangle |M_0\rangle$$  \hfill (51)

with $g = \int_{t_i-\epsilon}^{t_i+\epsilon} f(t)$. For small enough $g$ we can make the approximation

$$e^{-igASPM} \approx 1 - igASPM = U_i$$  \hfill (52)

We can calculate $U_iU_i^\dagger \approx 1$ to see that the state is normalized up to order $O(g^2)$.

Next is the post selection which is just the application of $|\phi\rangle \langle \phi|$ The state is now

$$\sqrt{P(\phi)} |SM_{ti}\rangle \approx |\phi\rangle \langle \phi | U_i |\psi\rangle |M_0\rangle$$

$$\approx \left[ \langle \phi | [1 - igASP] |\psi\rangle \right]^{SM} |\phi\rangle |M_0\rangle$$

$$= \left[ \langle \phi |\psi\rangle - ig \langle \phi | A^S |\psi\rangle P \right]^{SM} |\phi\rangle |M_0\rangle$$

$$= \langle \phi |\psi\rangle [1 - ig\{A\}]_w P^{SM} |\phi\rangle |M_0\rangle$$

where $P(\phi)$ is the probability for post selection. Before proceeding to the main result which involves a second approximation let us calculate $P(\phi)$.  \hfill (53)
\[
P(\phi) = \langle M_0 | \phi \langle \psi | 1 + ig \{ A \}_w P \rangle^M \langle \phi | \psi \rangle [1 - ig \{ A \}_w P]^M | \phi \rangle | M_0 \rangle \quad (57)
\]

\[
= | \langle \psi | \phi \rangle |^2 \langle M_0 | [1 + ig \{ A \}_w P] [1 - ig \{ A \}_w P] | M_0 \rangle \quad (58)
\]

\[
\approx | \langle \psi | \phi \rangle |^2 \langle M_0 | [1 + ig(\{ A \}_w - \{ A \}_w^\ast) P] | M_0 \rangle \quad (59)
\]

\[
= | \langle \psi | \phi \rangle |^2 [1 + g \text{Im}(\{ A \}_w) \langle M_0 | P | M_0 \rangle] \quad (60)
\]

We note that we can always choose the initial state of the meter \( |M_0 \rangle \) such that \( \langle M_0 | P | M_0 \rangle = 0 \) so that the probability for post selection is unchanged. Moreover if we expect the imaginary part of the weak value to vanish the meter state can be arbitrary and \( P(\phi) \) will not change.

This gives the first property of the weak measurement process

**Property 1.** The weak measurement interaction does not change the probability for post-selection.

We can now go back to our expression for the final state. Noting that the system and meter are in a product state we can trace out the system and leave the meter in a pure state.

\[
|\mathcal{M}_{ff} \rangle \approx [1 - ig \{ A \}_w P] | M_0 \rangle \quad (61)
\]

\[
\approx e^{ig \{ A \}_w P} | M_0 \rangle \quad (62)
\]

This produces the effective (non Hermitian) Hamiltonian (47). The outcome of our measurement is now an effective potential rather than a classical number.

**Property 2.** The result of a weak measurement is an the evolution of the meter according to effective potential proportional to the weak value. The effective Hamiltonian is (47) with 

\[
g = \int_{t_i-\epsilon}^{t_i+\epsilon} f(t)
\]

Note that the above statement will usually require \( g \{ A \}_w <<< 1 \) although in some specific cases that is not necessary. For example when \( |\psi \rangle \) or \( |\phi \rangle \) is an eigenstate of \( A \) there is no need for the weak measurement to be weak. In fact there is a more general statement we can make regarding deterministic strong values.

First let us define the above term: A two state vector \( |\phi \rangle \langle \psi | \) has a deterministic strong value \( a_i \) if for a strong measurement of \( A \), \( P(a_i) = 1 \). From the ABL formula we can see that this would happen if \( |\phi \rangle \) or \( |\psi \rangle \) are eigenstates of \( A \) but there are more general cases.

**Property 3.** For any two state vector
(a) \( p(a_i) = 1 \Rightarrow \{A\}_w = a_i \)

(b) If \( A \) is dichotomic then \( \{A\}_w = a_i \Leftrightarrow P(a_i) = 1 \)

(c) If \( a_i \) is a deterministic strong value then a von Neumann measurement will produce an effective potential of \( \{A\}_w = a_i \) for any interaction strength.

**Exercise 6.** Prove the above statement. Note that we already proved the second part

### X. PROPERTIES OF WEAK MEASUREMENTS AND WEAK VALUES

Weak values are the result of a weak measurement. They behave in many ways like expectation values in the ‘standard’ formalism. I already mentioned that when the pre and post selection coincide the weak value is the expectation value. Since the measurement is non-disturbing post-selecting the initial (pre selected) state is almost identical to making no-post selection. We can therefore make the following statement.

**Property 4.** For any observable \( A \), the average weak value of an ensemble of systems pre-selected in the state \( |\psi\rangle \) (but not post selected) is the same as the weak value of a system pre selected in \( |\psi\rangle \) and post selected in the same state \( |\psi\rangle \). This is just the expectation value \( \langle \psi | A | \psi \rangle \).

I will not prove this property, but note that the post interaction state is very close to \( |\psi\rangle e^{-ig<A>^P} |M_0\rangle \) with \( <A> = \langle \psi | A | \psi \rangle \), i.e the overlap is 1 up to terms of order \( g^2 \).

Another property which we already noted is the sum rule:

**Property 5.** The weak value of a sum of operators is the sum of the weak values \( \{A+B\}_w = \{A\}_w + \{B\}_w \).

However weak values are also very different from expectation values. We already noted the following property

**Property 6.** The weak value is an unbounded (but finite) complex number.

Expectation values on the other hand are real numbers bounded by the largest and smallest expectation values of the observable. Another deviation from expectation values (and classicality) is that weak don’t obey the product rule.

**The product rule:** If \([A,B] = 0 \) and \( a_i, b_j \) are the corresponding deterministic strong values for \( |\psi\rangle \) then \( \langle \psi | A | \psi \rangle \langle \psi | B | \psi \rangle = \langle \psi | AB | \psi \rangle = a_i b_j \)
Property 7. The product rule fails for weak values

This is apparent in the three box paradox.

Exercise 7. Show that the product rule fails for the three box paradox

A. Measuring weak values

As we noted the ‘result’ of a weak measurement is best interpreted as an effective potential. In an actual experiment, however we would like to extract the weak value, preferably using a weak measurement. I will show two methods of doing this, the first is a simple repetition of the experiment many times to improve the estimate of the weak value. The second is a slight modification of the first using a single measuring device and exploiting the fact that the weak measurement is a coherent process.

For simplicity I will again go back to using the qubit meter. So $P = \sigma_y$ $Q = \sigma_z$ and $|M_0\rangle = |+\rangle$.

1. Weak measurements on a large ensemble

For a single weak measurement the final state of the meter (after post selection) is given by (62) so

$$|M_f\rangle \approx |+\rangle - ig\{A\}_w\sigma_y |+\rangle = |+\rangle - g\{A\}_w |-\rangle$$

(63)

so

$$\langle M_f | \sigma_z | M_f \rangle = -2gRe(\{A\}_w)$$

(64)

and

$$\langle M_f | \sigma_y | M_f \rangle = -2gIm(\{A\}_w)$$

(65)

However the uncertainty in these variables is quite large due to the small value of $g$. To improve precision it is possible to repeat the experiment $N$ times. The standard deviation will then decrease as $1/\sqrt{N}$. Given standard deviation of order $g$ we would require $N = O(g^2)$ repetitions.
2. Weak measurements that are not really weak

It is possible (in theory\(^3\)) to consider \(N\) systems pre and post-selected in \(\langle \phi | \psi \rangle\). In such a scenario instead of using a meter to probe each system separately it is possible to use a single meter to probe all \(N\) systems [10].

Let us assume for simplicity that the weak value \(\{A\}_w\) is real so that a weak measurement rotates the meter around the \(Y\) axis (of the Bloch sphere), i.e \(|M\rangle \sigma_y |M\rangle \approx O(g^2)\) after the first weak measurement (see below).

The procedure is as follows. We weakly measure one system at a time with coupling \(g = k/N\) where \(0 < k < 2\pi\) is a constant and \(N\) is the number of systems pre and post selected in \(\langle \phi | \psi \rangle\). The interaction per each system is given by.

\[
\langle \phi|\psi \rangle U_{eff} = \langle \phi| e^{-i \frac{k}{N} A \sigma_y} |\psi \rangle = \langle \phi|\psi \rangle [e^{-i \frac{k}{N} \{A\}_w \sigma_y} + \frac{1}{N^2} \alpha \mathbb{I} + \frac{1}{N^2} \beta \sigma_y]
\] (66)

where \(\alpha\) and \(\beta\) are constants such that \(|\alpha| \ll N, |\beta| \ll N|\). (Note that we can always choose \(N\) to satisfy these conditions). For simplicity we define \(\hat{O} = \alpha \mathbb{I} + \beta \sigma_y\). We now repeat this coupling \(n \leq N\) times. The state of the meter will be

\[
\sqrt{P(\phi \otimes n)} |\phi\rangle = [e^{-i \frac{k}{N} \{A\}_w \sigma_y} + \frac{1}{N^2} \hat{O}]^n |+\rangle \approx [e^{-i \frac{k}{N} \{A\}_w \sigma_y} + \frac{n}{N^2} \hat{O} + ...] |+\rangle
\] (67)

where \(\hat{O} = e^{-i \frac{n-1}{N^2} \{A\}_w \sigma_y} \hat{O}\).

Here we can see two things. First the meter is still (to order \(n/N^2\)) in a state \(\langle \mathcal{M}_n | \sigma_y | \mathcal{M}_n \rangle \approx 0\) so that the weak measurement approximation will hold for the next iteration. Second at \(n = N\) we have a rotation of \(e^{-ik\{A\}_w \sigma_y}\) where \(k\) is not necessarily a small number.

3. Weak measurements of non-identical ensembles

The method introduced above does not actually require all the systems to be described by the same two state vector. It is enough that all systems in the ensemble have the same weak value for the relevant operator.

\(^3\) In practice the probability of such an event is \(|\langle \phi|\psi \rangle|^2 N\) so usually such a scenario is unlikely.
XI. QUANTUM CHESIRE CAT

To finish off this lecture I will describe the Chesire cat experiment [11]. The experiment involves a spin 1/2 particle with position \( L, R \) and spin +1, −1 degrees of freedom. It is pre-selected in the state

\[
|\psi\rangle = \frac{1}{2}([0] + [1])|R\rangle + ([0] − [1])|L\rangle
\]

and post selected in

\[
|\phi\rangle = \frac{1}{2}([0] + [1])[|R\rangle + |L\rangle]
\]

To calculate \( \sigma_z \) for this system we note that

\[
\langle \phi | 1 \rangle \langle 1 | \psi \rangle = \frac{1}{4}([R] + [L])[|R\rangle − |L\rangle] = 0
\]

So both ABL and weak measurements agree that \( < \sigma_z > = +1 \). Also to see where the particle can be found we use

\[
\langle \phi | L \rangle \langle L | \psi \rangle = \frac{1}{4}([0] + [1])[|0\rangle − |1\rangle] = 0
\]

So the particle is on the right.

We can now measure the spin on each arm using a weak measurement. A particle on the right has a spin state of \( |+\rangle \) so \( \{\sigma_z \otimes |R\rangle \langle R|\}_w = 0 \).

On the other hand a weak spin measurement on the left will give \( \{\sigma_z \otimes |L\rangle \langle L|\}_w = 1 \) so the particle’s spin shows up on the Left.

On the left we have a grin (spin +1) without a cat (particle).

**Part IV**

**Experiments and applications**

In this lecture I will go over 2-3 schemes related to weak measurement. I will start with Hardy’s paradox, next I will describe an experiment that uses weak value amplification to measure tiny changes in interaction strength, finally I will show how weak measurements can be used to break Wiesner’s quantum money scheme.
XII. HARDY’S PARADOX

Hardy’s paradox is an interesting variation on interaction free measurement. In this paradox an electron and positron can only reach the detector by going through a path where they must annihilate each other.

The easiest way to introduce the setup is via interaction free measurements.

A. Interaction free measurement

Elitzur and Vaidman [12] proposed the following question: A factory creates bombs that are activated using a single photon. Due to manufacturing imperfections some bombs do not work. In a working bomb a photon hitting the trigger will be absorbed and the bomb will explode. In a defective bomb a photon will reflect from the trigger unaffected. As a potential buyer you only want bombs that work, is there any way to test them?

Somewhat surprisingly the answer is yes. The original ‘test’ proposed by Elitzur and Vaidman will detonate a working bomb 50% of the time and will detect that it is working 25% of the time, the remaining 25% will yield an ‘inconclusive’ result. This efficiency of this test can be improved using the quantum Zeno effect. For our purposes it is enough to describe the original solution.

We with a balanced interferometer, a photon coming in at the top would end up at the bottom with certainty. The top right mirror of the interferometer is now replaced by the trigger (fig 6). If the bomb is faulty the interferometer is just a standard balanced interferometer and the photon would end up at the bottom detector \( I \). If the bomb works there is a 50% probability it will explode, however if it does not explode the photon will have equal probability to hit \( I \) or \( W \), the bottom or right detectors respectively.

B. Hardy’s interferometer

In Hardy’s setup [13] the photon interferometer is replaced by a positron and an electron interferometer (fig 7. The interferometers are setup in such a way that if the electron goes on the top arm and the positron on the bottom arm they will overlap and annihilate each-other.
FIG. 6: The bomb test. A working bomb has only 50% chance to explode and 25% chance to be detected on detector $W$ or $I$. A reading on $W$ indicates a working bomb while a reading on $I$ is inconclusive.

The state of the system at the mirrors is

$$|EP_m⟩ = \frac{1}{2} |TE, TP⟩ + |BE, BP⟩ - |TE, BP⟩ - |γ⟩⟩$$ (72)

At the detectors we have

$$|EP_D⟩ = \frac{1}{4} | - |WE, WP⟩ + 3 |IE, IP⟩ + |WE, IP⟩ - |IE, WP⟩ - 2 |γ⟩⟩$$ (73)

Exercise 8. Derive expression (73)

We can see that it is possible to detect the pair at $WE, WP$ with $P(we, wp) = 1/16$. Let us now post select on this case. First note that a detection at $WE$ implies the positron must have taken the top arm (acted like a working bomb). Likewise $WP$ implies to electron took the bottom arm. However both events occurring simultaneously lead to annihilation (a photon) so we have a contradiction.

For a full analysis let us look at the backward evolving state in the mirrors (in this case annihilation occurs at a different place in the circuit).

$$⟨EP_m⟩ = \frac{1}{2} [⟨TE, TP⟩ + ⟨BE, BP⟩ + ⟨TE, BP⟩ + ⟨γ^∗⟩]$$ (74)
Looking at the strong values we get

\[
P(TE) = 0 \quad P(BE) = 1 \quad (75)
\]
\[
P(BP) = 0 \quad P(TP) = 1 \quad (76)
\]
\[
P(BE, TP) = 0 \quad P(TE, TP) = 1 \quad P(BE, BP) = 1 \quad (77)
\]

Weak values give us \([14, 15]\)

\[
\{TE\}_w = 0 \quad \{BE\}_w = 1 \quad (78)
\]
\[
\{BP\}_w = 0 \quad \{TP\}_w = 1 \quad (79)
\]
\[
\{BE, TP\}_w = 0 \quad \{TE, TP\}_w = 1 \quad (80)
\]
\[
\{BE, BP\}_w = 1 \quad \{TE, BP\}_w = -1 \quad (81)
\]

XIII. WEAK VALUE AMPLIFICATION\([16]\)

As we already noted weak values can be much larger than eigenvalues. Can this be used for some type of amplification? Again the answer is no but...

First let me describe an experiment involving a photon with two relevant degrees of freedom, Position \(Q\) (with conjugate momentum \(P\)) and polarization. Using the operator

\[
S = |H\rangle \langle H| - |V\rangle \langle V|
\]

we can define the effect of a birefringent crystal via the Hamiltonian

\[
H = vSQ \quad (82)
\]

with \(v\) the displacement speed. The unitary evolution depends on the time \(\tau\) spent in the crystal but will usually be small

\[
U = e^{-i\epsilon SP} \quad (83)
\]

For small \(\epsilon = v\tau\) this interaction is weak. We design the experiment such that the initial state going into the crystal is

\[
|\psi\rangle |Q = 0\rangle \quad (84)
\]

after the crystal we place a polarizer that lets through polarization \(|\phi\rangle\) and later we place a detector that detects the beam’s position \(Q\). The shift in \(Q\) is (assuming \(\epsilon\) is small enough) proportional to \(\epsilon \{S\}_w\). So that we ‘amplified’ the interaction by the weak value. Choosing
a large weak value allows us to determine $\epsilon$ with high precision. This will however come at a cost. The number of photons reaching the detector will be low.

Is this a good strategy for measuring $\epsilon$. From a purely theoretical point of view the answer is no [17]. But from a practical point this may be useful. In this particular situation
the light source can produce a very large number of events, however the (expensive) detector can only detect a small number. The weak value technique is a clever way of placing the a filter to reduce the number of photons arriving at the detector.

**XIV. ZENO TYPE PROTECTIVE MEASUREMENTS**

I will now show how weak measurements can be used to ‘clone’ a quantum state under very special circumstances. Consider the following game.

Alice gives Bob a spin 1/2 particle in one of four states $|0\rangle, |1\rangle, |+\rangle, |-\rangle$ and tells Bob he is not allowed to touch the particle. To make sure she measures the particle often using a projective measurement. If the result is the original state $|\psi\rangle$ she is satisfied that Bob did not touch the particle, if it is not, she fires Bob.

Bob wants to know the spin state of the particle and decides to use a weak measurement technique to fool Alice. He uses his own spin 1/2 particle (initially in the state $|0\rangle^B$) as the meter. The protocol is carried out in two stages, at the first stage he determines the basis using the following interaction Hamiltonian

$$H_i = f(t) |0\rangle \langle 0|^A \sigma_y^B$$

so that the interaction is

$$H_i = e^{-ig|0\rangle\langle 0|^A \sigma_y^B}$$

Each time Alice makes a measurement he switches the interaction on again. He does this $\pi/g$ times. There are two possible cases to be analyzed, in the first case $|\psi\rangle$ is in the $|0\rangle, |1\rangle$ basis so the measurements don’t do anything and the phases $g$ add up to $\pi$ so that the state is back to $|0\rangle$.

In the second case $|\psi\rangle = |\pm\rangle$ we will first look at the first step (before the measurement so $t = 0.5$)

$$|SM_{0.5}\rangle = U |\psi\rangle |0\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle e^{i g \sigma_y} \pm |1\rangle \mathbb{I} \right) |0\rangle$$

Alice’s measurement of $|\pm\rangle$ has a probability of $P = 1 - O(g^2)$. After this measurement the state of the meter is

$$|SM_1\rangle = U |\psi\rangle |0\rangle = \frac{1}{2} [e^{i g \sigma_y} + \mathbb{I}] |0\rangle = e^{i \frac{g}{2} \sigma_y} |0\rangle$$
The same logic follows for every step such that the probability of success is

$$P(\text{no detection}) = [1 - O(g^2)]^{\pi/g} \approx [1 - O(g)]$$

and the final state is $e^{i\pi/2\sigma_y}|0\rangle = |1\rangle$

Bob can now with certainty discover the Basis. At this stage he just needs to make the correct measurement.

XV. REFERENCES


